M-math 1st year, final exam, May 7th 2012
Subject : Functional Analysis

Time : 3 hours Max.Marks 100.

1. Let $(H, <.,.>)$ be a Hilbert space. Let $A \in \mathbf{B}(H)$. Prove that :
a) $Ker A^* = (range\ A)^\perp$.
b) $Ker(A^*A) = Ker A$.
c) If dim(H) $< \infty$, then A, A^*A, $\sqrt{(A^*A)}$ have the same rank, i.e., the dimensions of their ranges are the same.
d) $(Ker A)^\perp =$ the closure of range $A^* = \overline{range\ A^*}$. (3+4+4+4)

2. Let $(H, <.,.>)$ be a Hilbert space. Let $A \in \mathbf{B}(H)$.
a) Show that if A is normal then $spr(A) = \|A\|$.
b) If A is self adjoint and $A^n = 0$ for some $n \geq 1$, then $A = 0$. (7+3)

3. Show that a Banach space X is reflexive iff its dual X^* is reflexive. (6+9)

4. a) Show that every Banach space X is isometrically isomorphic to a closed linear subspace of $C(E)$ where E is a compact Hausdorff space.

b) Let $X = C[0,1]$, (continuous real valued functions on [0,1]) with the unform norm $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$. Its dual X^* may be identified as $X^* \cong NBV[0,1] := \{g : [0,1] \to \mathbb{R}, g(0) = 0, g$ of bounded variation and right continuous on$[0,1]\}$, with the total variation norm. Show that if $g \in X^*$ and $g(x) \geq 0, x \in X$, then g is given by a non decreasing function on $[0,1]$. (Hint : evaluate $g(x)$, where x is the indicator function of (s,t], by choosing an appropriate sequence $x_n \in X$ decreasing to $x \in X$.)

c) If $\{\mu_n\ n \geq 1\}$ is a sequence of probability measures on $[0,1]$, show that there exists a subsequence $\{n_k\} \subset \{n\}$ and a probability measure μ on $[0,1]$ such that

$$\int_0^1 f(t)\, d\mu_{n_k}(t) \to \int_0^1 f(t)\, d\mu(t), \quad \forall f \in C[0,1].$$

(Hint : identify μ_n as an element of X^*, where X as in b).) (7+6+7)

5. Let X be a Banach space and $A \in \mathbf{B}(X)$. For $t \geq 0$, define $S_t := exp(tA)$. Show that $S_t \in \mathbf{B}(X)$, $S_{t_1+t_2} = S_{t_1} \circ S_{t_2}$, and for all $x \in X$, $\lim_{t \to 0} S_t x = x$. (10)

6. Let $X = l_2$ and $\alpha = (\alpha_n) \in l_\infty$. Then show that the diagonal operator $A_\alpha : l_2 \to l_2$, defined as $A_\alpha x := (\alpha_n x_n)$ for $x = (x_n) \in l_2$ is compact iff $\alpha_n \to 0$. (10)

7. Let $H := L^2[0, 1]$, (with Lebesgue measure). For $\phi \in L^\infty$, let $M_\phi : H \to H$. Show that $\|M_\phi\| = \|\phi\|_\infty$. (10)

8. Let $H = L^2[-1, 1]$ (with Lebesgue measure), $A : H \to H, Af(x) := x^2 f(x)$. Then $A \geq 0, A \in B(H)$.

a) Show that A has no cyclic vector in H. (Hint : Given $f \neq 0 \in H$, construct $g = g(f) \neq 0 \in H$, such that $< g, A^n f >_H = 0 \ \forall n \geq 1$.)

b) Let H_e and H_o be the closed subspaces of H consisting respectively of the even and odd functions in H. Show that H_e and H_o are orthogonal subspaces of H and each is a cyclic subspace for A. (6+9)